Exam / Numerical Analysis / April 10th 2025, University of Groningen

Instructions

- Use a pen (blue or black ink) to write your answers.
- The grade will be computed as the number of obtained points, plus 1.
- You have 2 hours to complete the exam. When applicable, people with special facilities have 2h20 minutes in total.
- **Split your answers into two sheets: (a)+(b) and (c)+(d).**

Questions

Consider the scalar ODEs:

$$y'(t) = f(t, y(t)), \quad y(0) = y_0 \neq 0, \tag{1}$$

with f continuous in t and Lipschitz-continuous in y.

Consider now the following discretization method: Find u_{n+1} such that

$$u_{n+1} = u_n + hf(t_{n+\beta}, u_{n+\beta}), \quad n \geq 0, \quad u_0 = y_0 \tag{2}$$

with $x_{n+\beta} := (1-\beta)x_n + \beta x_{n+1}$.

(a) [2.5] Assume that a solution to Equation (2) exists. Show that there is a unique solution for u_{n+1} for a small enough step size h.

Let us assume that there are two solutions for u_{n+1}, say x and y **(0.5 pt)**. Then, we can write Equation (2) for each of them **(0.5 pt)**:

$$x = u_n + hf(t_{n+\beta}, (1-\beta)u_n + \beta x)$$
$$y = u_n + hf(t_{n+\beta}, (1-\beta)u_n + \beta y)$$

and hence **(0.5 pt)**

$$\begin{aligned} |x-y| &= |hf(t_{n+\beta}, (1-\beta)u_n + \beta x) - hf(t_{n+\beta}, (1-\beta)u_n + \beta y)| \\ &\leq hL|\beta(x-y)| = hL\beta|x-y| \end{aligned}$$

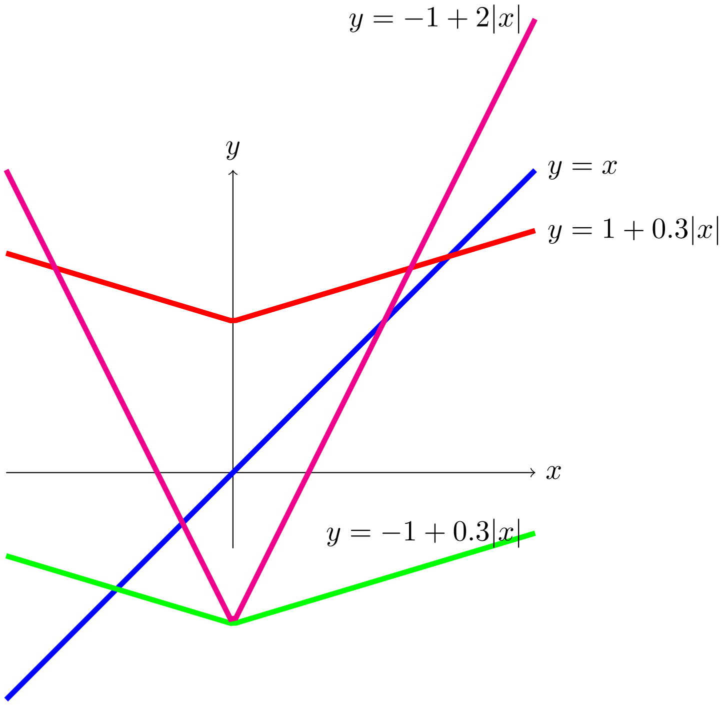
where we have used the Lipschitz continuity of f in the second step. Then, if we choose $h < 1/(L\beta)$ **(0.5 pt)**, we have that $|x-y| < |x-y|$ which is a contradiction, and hence there is a unique solution for u_{n+1} **(0.5 pt)**.

(b) [2] For $f(t, y) = \lambda|y|$, with $\lambda > 0$, find the specific value ranges of h that lead to a unique solution of Equation (2). Assume that $\beta = 1$ for simplicity. Analyze the cases $u_n > 0$, $u_n = 0$ and $u_n < 0$ separately (hint: drawing could be helpful).

We can rewrite Equation (2) as follows **(0.25 pt)**:

$$x = u_n + h\lambda|x|$$

We can now graphically represent the Equation (2) as the intersection of $y = x$ and $y = u_n + h\lambda|x|$ **(0.5 pt)**. For the case $u_n = 0$, the intersection is unique for every h and it is given by $x = 0$ **(0.25 pt)**. For the case $u_n > 0$, the red curve shows that if $h < 1/\lambda$, then there is a unique intersection at some $x > 0$ **(0.25 pt)**, and there is no intersection otherwise **(0.25 pt)**. For the case $u_n < 0$, in the green curve we note that if $h \leq 1/\lambda$, then there is a unique intersection for a negative x **(0.25 pt)**, and there are two intersections if $h > 1/\lambda$, one positive and one negative **(0.25 pt)**. Hence, for both cases, we need $h < 1/\lambda$ to obtain a unique solution using the beta-method.



(c) $\boxed{1}$ Comment on which root-finding methods should and should not be used to numerically approximate the solution of Equation (2) when $f(t, y) = \lambda|y|$, for h such that the the solution is unique.

To solve the problem numerically, the Newton method is not guaranteed to converge to the solution, because the function is not differentiable at $x = 0$ **(0.5 pt)**.
For the method that could work, there are a few alternatives **(0.5 pt)**:

- We can solve the solution directly by splitting in the cases $x > 0$, $x = 0$ and $x < 0$, as it is linear in this way, and check which solution satisfies the original equation.
- We could use a fixed-point iteration method, of the type $\phi(x) = x - \alpha g(x)$, with $g(x) = x - u_n - h\lambda|x|$ and $\alpha > 0$ a step size small enough to guarantee the convergence of the method (we can find such α because g is a contraction).
- We can use a bisection method to find the root of the problem, which is guaranteed to converge because the function is continuous and we can find an interval where it changes sign.

(d) $\boxed{3.5}$ Show that the first derivative of the Newton iteration function to solve Equation (2) evaluated at u_n is **bounded by 1**, for h small enough.

Assume that the absolute value of all derivatives of f with respect to y are bounded and that

$$\left.\frac{\partial f(t_{n+\beta}, y)}{\partial y}\right|_{y=u_n} \leq 0. \tag{3}$$

What's the practical implication of this result in the numerical approximation of the solution of Equation (2)?

The Newton iteration function is given by **(0.25 pt)**

$$G(x) = x - \frac{r(x)}{r'(x)}$$

with **(0.25 pt)**

$$r(x) = -x + u_n + hf(t_{n+\beta}, (1-\beta)u_n + \beta x)$$

and **(0.25 pt)**

$$r'(x) = -1 + h\beta \frac{\partial f(t_{n+\beta}, y)}{\partial y}\bigg|_{y=(1-\beta)u_n+\beta x}$$

Then, we can compute the derivative of G as follows **(1 pt)**:

$$\begin{aligned}
G'(x) &= 1 - \frac{r'(x)r'(x) - r(x)r''(x)}{(r'(x))^2} = \frac{r(x)r''(x)}{(r'(x))^2} \\
&= \frac{(-x + u_n + hf(t_{n+\beta}, (1-\beta)u_n + \beta x))(h\beta^2 \frac{\partial^2 f(t_{n+\beta},y)}{\partial y^2}\Big|_{y=(1-\beta)u_n+\beta x})}{-1 + h\beta \frac{\partial f(t_{n+\beta},y)}{\partial y}\Big|_{y=(1-\beta)u_n+\beta x}}
\end{aligned}$$

Evaluating at the solution of the previous time step, i.e. $x = u_n$ **(0.75 pt)**, we have

$$\begin{aligned}
G'(u_n) &= \frac{h^2\beta^2 f(t_{n+\beta}, u_n) \frac{\partial^2 f(t_{n+\beta},y)}{\partial y^2}\Big|_{y=u_n}}{\left(-1 + h\beta \frac{\partial f(t_{n+\beta},y)}{\partial y}\Big|_{y=u_n}\right)^2} \\
&< \frac{h^2\beta^2 \left| f(t_{n+\beta}, u_n) \frac{\partial^2 f(t_{n+\beta},y)}{\partial y^2}\Big|_{y=u_n} \right|}{\left| -1 + h\beta \frac{\partial f(t_{n+\beta},y)}{\partial y}\Big|_{y=u_n} \right|^2} \\
&\leq h^2\beta^2 M_0 M_2
\end{aligned}$$

where the denominator is bounded by one due to the assumption in Equation (3), and we have used the bounds on the zero- and second-order derivatives of f in the last step. Hence, $G'(u_n)$ is bounded by 1 if **(0.5 pt)**

$$h < \frac{1}{\beta\sqrt{M_0 M_2}}$$

The implication of this result is that there always exists one $h > 0$ such that the Newton method converges to the solution of Equation (2) for the initial guess equal to the solution of the previous time step **(0.5 pt)**.